

Lower bounds on the entanglement needed to play XOR non-local games

William Slofstra *

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Abstract

We give an explicit family of XOR games with $O(n)$ -bit questions requiring 2^n ebits to play near-optimally. More generally we introduce a new technique for proving lower bounds on the amount of entanglement required by an XOR game: we show that near-optimal strategies for an XOR game G correspond to approximate representations of a certain C^* -algebra associated to G . Our results extend an earlier theorem of Tsirelson characterising the set of quantum strategies which implement extremal quantum correlations.

1 Introduction

The purpose of this paper is to study the amount of entanglement required to play an XOR non-local game optimally or near-optimally. In a non-local game, Alice and Bob are asked questions chosen at random according to a known distribution. They win if their answers match the answers required by the game's rules, and lose otherwise. They are physically separated and unable to communicate during the game, so they cannot hope to win every game. Instead, they must meet in advance to determine the best strategy, i.e. one that maximizes the probability that they win. Bell's celebrated theorem [2] states that for some games Alice and Bob can increase their probability of winning if they employ a quantum strategy, in which they make use of an entangled quantum state.

From the viewpoint of quantum computation, a quantum strategy is a means of performing a distributed computation with a higher success probability than what can be achieved classically with no communication [4] [8]. From this viewpoint it is natural to want to know how much entanglement is required to implement an optimal quantum strategy. Entanglement of a pure state can be measured using either the entropy of entanglement (measured in ebits), or the dimension of the underlying Hilbert space at each site; the two measures are related by the fact that any state in $\mathbb{C}^d \times \mathbb{C}^d$ has at most $\log_2 d$ ebits. In physics, the importance of lower bounds stems from the fact that non-local games provide empirical evidence that a quantum system is entangled [20]. Lower bounds on the entanglement required by a game imply that the game can be used to verify the degree of entanglement in a quantum system, at least in principle [6].

In this paper we focus on a subset of non-local games generalizing the well-known Clauser-Horne-Shimony-Holt (CHSH) game [7]: Alice and Bob each receive exactly one question (drawn from finite sets S and T respectively), each question requires a one-bit answer, and the correct response depends only on the XOR of Alice's and Bob's answers. Non-local games meeting these restrictions are known as XOR non-local games. They have attracted interest in part because of the influential results of Tsirelson [17] [18] which guarantee the existence of an optimal strategy

*Department of Mathematics, University of California, Berkeley. Email: slofstra@math.berkeley.edu

supported on a finite-dimensional Hilbert space, and make it possible to find an optimal strategy using semidefinite programming methods. Tsirelson also proved a lower bound on entanglement for quantum strategies by showing that strategies which implement extreme points of the set of quantum correlations require dimension exponential in the rank of the correlation matrix (see Theorem 2.7).

Tsirelson's lower bound can be dualized (see Proposition 2.6) to show that there are many XOR games requiring dimension exponential in the size of the question sets. In particular non-local games exist which are able to verify that quantum states have arbitrarily high dimension. More recent work has readdressed this issue with different methods. Junge, Palazuelos, Perez-Garcia, Villanueva, and Wolf study the ratio $\epsilon(G, m)/\epsilon(G, n)$ for general two and three player non-local games, where $\epsilon(G, d)$ is the maximum success bias for G achievable with a quantum system of dimension d [15] [11]. In particular, they show that two-player games exist for which this ratio is arbitrarily high. However, question sets of size $O(2^{d \log^2 d})$ are required to verify dimension d . Briet, Buhrman, and Toner study a similar ratio for XOR games, where the parameter d in $\epsilon(G, d)$ refers to the rank of the quantum correlation matrix rather than the dimension of the quantum system. The ratio $\epsilon(G, m)/\epsilon(G, n)$ cannot be arbitrarily high for XOR games, since Tsirelson showed that $\epsilon(G, d)/\epsilon(G, 1)$ is always bounded by Grothendieck's constant. Nonetheless, Briet, Buhrman, and Toner show non-constructively that there are XOR games for which this ratio is greater than $1 + 1/2m + 1/2n - O(1/m^2)$ [5]. Vertesi and Pal give an explicit family of XOR games with question sets of size $O(n^2)$ requiring dimension n to play optimally [19]. Their lower bound is proved by giving a lower bound on the rank of the optimal correlation matrix.

In contrast to the body of work on lower bounds, little is known about upper bounds on entanglement for general non-local games. Tsirelson proved that XOR games with question sets of size m and n require at most $\lfloor r/2 \rfloor$ ebits, where r is the largest integer such that $\binom{r}{2} < m + n$. One purpose of this paper is to point out that this lower bound is tight, so XOR games with small question sets can require high entanglement to play optimally. This can be proved by applying Tsirelson's lower bound method to specific families of games, such as the family of examples due to Vertesi and Pal. For completeness, we give another family of examples for which Tsirelson's bound is tight in Section 7. Using a new lower bound technique, we are also able to study the amount of entanglement required by near-optimal strategies. Cleve, Hoyer, Toner, and Watrous showed that an $m \times n$ XOR non-local game has an ϵ -optimal quantum strategy on a Hilbert space of dimension $(m + n + 1)^{O(\epsilon^{-2})}$ [8]. Their proof can be modified to be independent of the size of the question sets, so that fixing ϵ bounds the entanglement cost for all XOR games (this was apparently first observed by O. Regev, and the author is grateful to R. Cleve and R. Jain for a proof [10]). We show, for a specific family of games, that ϵ -optimal strategies require a Hilbert space of dimension $\min(O(\epsilon^{-1/12}), 2^{\lfloor r/2 \rfloor})$, where r is the integer from Tsirelson's upper bound.

The main idea of this paper is to associate to each XOR non-local game a finitely presented C^* -algebra \mathcal{A} with the property that (a) optimal strategies correspond to representations of \mathcal{A} , and (b) near-optimal strategies correspond to approximate representations of \mathcal{A} . The presentation for \mathcal{A} depends only on the game rules and a small amount of additional information: the game's marginal biases. Inspiration for this idea comes from the work of Summers and Werner [16] and Tsirelson [17] [18]. Summers and Werner show that the Clifford algebra of rank two satisfies property (a) above for the CHSH game. The main statement of Tsirelson's lower bound is that if an XOR non-local game has a unique optimal correlation matrix C then the algebra generated by the observables of any optimal non-degenerate quantum strategy is isomorphic to the Clifford algebra with rank C generators. Our approach is more general, in that it applies to all XOR games, and also to near-optimal strategies.

The paper is organized as follows. In the next section we give an overview of our results, including the necessary background material and the relation to Tsirelson's results. Marginal biases are introduced in Section 3, and the main result for optimal strategies follows in Section 4. A link between near-optimal strategies and approximate representations is established in Section 5. In Section 6 we show that the Clifford algebra is stable. Finally, explicit examples are given in Section 7.

2 Definitions and statement of results

2.1 XOR non-local games and quantum strategies

The rules for an XOR non-local game are comprised of two finite sets S and T of questions, a probability distribution π on $S \times T$, and a function $f : S \times T \rightarrow \{0, 1\}$ recording the correct answers. Alice and Bob have access to the rules and may communicate before the game begins. When the game begins, a pair of questions (i, j) is chosen with probability $\pi(i, j)$. Alice receives question i , and must output a bit a ; Bob receives question j and outputs a bit b . Alice and Bob win if $a \oplus b$ (the XOR of the two bits) matches $f(i, j)$. All that matters about S and T is their size, so we assume $S = \{1, \dots, m\}$ and $T = \{1, \dots, n\}$. The cost matrix for this game is defined to be the $m \times n$ matrix G with $G_{st} = (-1)^{f(i,j)} \pi(i, j)$. The value of $\pi(i, j)$ can always be recovered from G_{ij} , and $f(i, j)$ can also be recovered if $\pi(i, j) \neq 0$. If $\pi(i, j) = 0$ then $f(i, j)$ is irrelevant to the game, so an $m \times n$ XOR non-local game is completely described by its cost matrix G . A matrix G is the cost matrix of an XOR non-local game if $\sum |G_{ij}| = 1$.

Let a_i be the binary random variable corresponding to Alice's output on input i , and let b_j be the binary random variable corresponding to Bob's output on input j . Then $(-1)^{f(i,j)}(-1)^{a_i}(-1)^{b_j}$ is 1 if Alice and Bob win on question i, j , and -1 otherwise. The sum

$$\sum_{i,j} G_{ij} (-1)^{a_i} (-1)^{b_j}$$

is the *expected success bias* for this behaviour, analagous to the expected success probability, but normalized between -1 and 1 .¹ In the quantum setting, Alice and Bob are allowed to select a shared bipartite quantum state $|\psi\rangle \in H_1 \otimes H_2$. In addition to the state $|\psi\rangle$, a quantum strategy consists of two families of observables $\{A_i : i \in S\}$ and $\{B_j : j \in T\}$, on the Hilbert spaces H_1 and H_2 respectively, with spectra contained in $\{-1, 1\}$. The success bias for this quantum strategy is

$$\sum G_{ij} \langle \psi | A_i \otimes B_j | \psi \rangle,$$

and the *quantum success bias* $\varepsilon_q(G)$ for the game is the maximum success bias across all quantum strategies. The amount of entanglement used by a quantum strategy will be measured using the entropy of entanglement of the state $|\psi\rangle$, which is by definition the von Neumann entropy of the partial trace of $|\psi\rangle$ with respect to H_1 . The dimension of a quantum strategy is $\min(\dim H_1, \dim H_2)$.

2.2 C^* algebras generated by self-adjoint indeterminates

Let R_1, \dots, R_k be non-commutative polynomials in indeterminates X_1, \dots, X_n . Let \mathcal{A} be a C^* -algebra generated by self-adjoint operators X_1, \dots, X_n . Then \mathcal{A} is said to be the universal C^* -algebra generated by self-adjoint indeterminates X_i satisfying relations R_1, \dots, R_k if for all Hilbert spaces H the map $\rho \mapsto (\rho(X_1), \dots, \rho(X_n))$ is a bijection between the representations ρ of \mathcal{A} on

¹If p is the success probability, then the success bias is $2p - 1$.

H and tuples (B_1, \dots, B_n) of bounded self-adjoint operators on H such that $R_i(B_1, \dots, B_n) = 0$ for all $i = 1, \dots, k$. The universal C^* -algebra for a set of relations is determined uniquely up to isomorphism, if it exists. The set of relations determining \mathcal{A} is not unique. Since we are interested in C^* -algebras given by certain relations, we will regard the generators and defining relations as part of the data of the algebra. A universal C^* -algebra for a given set of relations exists if and only if there are constants b_1, \dots, b_n such that $\|B_i\| \leq b_i$ whenever (B_1, \dots, B_n) is a tuple of operators satisfying the given relations. For convenience we introduce a slight generalization of the usual notion of a cyclic representation:

Definition 2.1. *Let \mathcal{A} be a C^* -algebra generated by self-adjoint indeterminates X_i satisfying relations R_j . A density-matrix representation of \mathcal{A} is an action of \mathcal{A} on a Hilbert space H , determined by a collection of self-adjoint operators B_i , along with a density matrix ρ on H such that $\mathcal{A}\rho H$ is dense in H .*

The following universal C^* -algebra will be used in examples, and in describing previous results of Tsirelson.

Definition 2.2. *The Clifford algebra of rank r is the universal C^* -algebra \mathcal{C}_r generated by indeterminates X_1, \dots, X_r satisfying the relations $X_i^2 = \mathbb{1}$ for all $1 \leq i \leq n$ and $X_i X_j = -X_j X_i$ for all $i \neq j$.*

A universal C^* -algebra generated by Y_1, \dots, Y_n is *Clifford* if for all $1 \leq i, j \leq n$ the anti-commutator $Y_i Y_j + Y_j Y_i$ is a scalar multiple of the identity. If \mathcal{A} is Clifford with $Y_i Y_j + Y_j Y_i = C_{ij} \mathbb{1}$ then $\mathcal{A} \cong \mathcal{C}_r$, where r is the rank of the matrix $\{C_{ij}\}$. The isomorphism can be chosen so that every Y_i corresponds to a linear combination of the distinguished generators X_j of \mathcal{C}_r . We will call a universal C^* -algebra *strongly Clifford* if the Clifford relations $Y_i Y_j + Y_j Y_i - V_{ij} \mathbb{1}$ are linear combinations of the defining relations. The representation theory of \mathcal{C}_r is well-known: \mathcal{C}_r has either one or two irreducible representations of dimension $2^{\lfloor r/2 \rfloor}$.

2.3 Main result: algebraic characterization of optimal solutions

If $\{A_i\}, \{B_j\}, |\psi\rangle$ is a quantum strategy on a bipartite Hilbert space $H_1 \otimes H_2$, define the *marginal (strategy)* on H_2 to be the collection of operators $\{B_j\}$, as well as the density operator ρ which is the partial trace of $|\psi\rangle$ with respect to H_1 . Note that the entanglement of the original strategy is the von Neumann entropy of ρ . A strategy $\{A_i\}, \{B_j\}, |\psi\rangle$ is said to be non-degenerate if there is no projection P commuting with all A_i such that $(P \otimes I)|\psi\rangle = |\psi\rangle$, and no projection Q commuting with all B_j such that $(I \otimes Q)|\psi\rangle = |\psi\rangle$. Every degenerate strategy projects down to a unique non-degenerate strategy.

Our main result is a precise description of the marginal strategies corresponding to optimal non-degenerate quantum strategies of a given XOR non-local game G . A key part of this description is the fact that for every $m \times n$ XOR non-local game G and $1 \leq i \leq m$ there is a constant c_i such that $\sum_j G_{ij} \langle \psi | A_i \otimes B_j | \psi \rangle = c_i$ whenever $\{A_i\}, \{B_j\}, |\psi\rangle$ is an optimal quantum strategy. We refer to c_i as the *i th marginal row bias for G* . The existence of the marginal biases is proved in Section 3.

If G is an $m \times n$ XOR non-local game with marginal row biases c_i , define the *solution algebra \mathcal{A} for G* to be the universal C^* algebra generated by self-adjoint indeterminates X_1, \dots, X_n , satisfying the relations

$$X_j^2 = \mathbb{1} \text{ for all } 1 \leq j \leq n, \text{ and } \left(\sum_j G_{ij} X_j \right)^2 = c_i^2 \cdot \mathbb{1} \text{ for all } 1 \leq i \leq m.$$

Theorem 2.3. *Let G be a XOR non-local game with no zero rows or columns and solution algebra \mathcal{A} . A collection of bounded linear operators $\{B_j : j \in T\}$ and density operator ρ on a Hilbert space H is the marginal of a non-degenerate optimal strategy for G if and only if the map $X_j \mapsto B_j$ induces a density-matrix representation of \mathcal{A} on H with density matrix ρ , and ρ commutes with the image of \mathcal{A} .*

Theorem 2.3 determines the entanglement required by G in the following sense:

Corollary 2.4. *Let G be an XOR non-local game with no zero rows or columns, and let \mathcal{A} be the corresponding solution algebra. Let N be the minimum dimension among non-zero representations of \mathcal{A} . Then the minimum entanglement used by an optimal quantum strategy for G is $\log_2 N$. The strategies which attain the minimum entanglement are the irreducible representations of \mathcal{A} of dimension N , with cyclic state I/N , where I denotes the identity operator.*

The dimension of a representation is the dimension of the underlying Hilbert space. Although the results above are stated for pure states, Corollary 2.4 extends trivially to mixed states when the entanglement of formation is used as an entanglement measure. A mixed state of minimum entanglement must be a mixture of maximally entangled states. The proofs of Theorem 2.3 and Corollary 2.4 are given in Section 4.

2.4 Comparison to Tsirelson's results

An $m \times n$ matrix $\{c_{ij}\}$ is called a *quantum correlation* if there is a quantum strategy $\{A_i\}, \{B_j\}, |\psi\rangle$ such that $c_{ij} = \langle \psi | A_i \otimes B_j | \psi \rangle$ for all i, j . The set C_{mn} of all $m \times n$ quantum correlations is closed and convex, so the problem of finding the quantum success bias for a game G can be formulated as the following convex programming problem:

$$(\Phi) \quad \max \sum_{i,j} G_{ij} c_{ij} \quad : \quad \text{where } \{c_{ij}\} \text{ is a quantum correlation.}$$

The optimisation problem (Φ) can be formulated as a semidefinite programming problem using the following theorem. This makes it possible to find a near optimal quantum correlation in practice.

Theorem 2.5 (Tsirelson, 1987). *An $m \times n$ matrix $\{c_{ij}\}$ is a quantum correlation matrix if and only if there are two families of vectors $\{u_i \in B(\mathbb{R}^N) : i \in S\}$ and $\{v_j \in B(\mathbb{R}^N) : j \in T\}$ such that $c_{ij} = u_i \cdot v_j$.*

Here $B(\mathbb{R}^N)$ denotes the unit ball in \mathbb{R}^N . Define a *vector strategy* for $\{c_{ij}\}$ to be two collections of vectors $\{u_i\}$ and $\{v_j\}$ in $B(\mathbb{R}^N)$ such that $c_{ij} = u_i \cdot v_j$. The proof of Theorem 2.5 is constructive, associating to a vector strategy a quantum strategy $\{A_i\}, \{B_j\}, |\psi\rangle$ with the same correlation. The observables $\{A_i\}$ and $\{B_j\}$ are constructed using the Clifford algebra C_N , and consequently Theorem 2.5 implies that every $m \times n$ game has an optimal quantum strategy using $\lfloor n/2 \rfloor$ ebits. Tsirelson [17] showed that this upper bound could be improved to $\lfloor r/2 \rfloor$ ebits, where r is the largest integer such that $\binom{r+1}{2} < m + n$.

Proposition 2.6. *For every $n \geq 2$ there is an $m \times n$ XOR game for which Tsirelson's bound is tight.*

Proof. Theorem 2.22 of [18] can be used to find a maximal rank extreme point of the set of quantum correlations. Any supporting hyperplane for this extreme point will satisfy the proposition.

For a constructive proof, apply either Theorem 2.3 or Theorem 2.7 below to explicit examples, such as the family of games in Section 7 ($m = \binom{n}{2}$) or the similar family of games in [19] ($m = \binom{n}{2} + 1$). \square

There is a strong relation between Theorem 2.3 and the following theorem of Tsirelson.

Theorem 2.7 (Tsirelson [17]). *Suppose that $\{c_{ij}\}$ is an extreme point of C_{mn} of rank r , and $\{A_i\}$, $\{B_j\}$, $|\psi\rangle$ is a non-degenerate strategy² representing $\{c_{ij}\}$. Then the strategy $\{A_i\}$, $\{B_j\}$, $|\psi\rangle$ is Clifford, and $|\psi\rangle$ uses at least $\lfloor r/2 \rfloor$ ebits.*

One way of understanding Tsirelson's theorem is that it associates a Clifford algebra to every extreme point of C_{mn} . XOR non-local games are in one-to-one correspondence with supporting hyperplanes of C_{mn} via the cost matrix, so Theorem 2.3 describes how to associate a C^* -algebra to every supporting hyperplane of C_{mn} . Thus the two results are complementary, and correspond to two approaches to describing convex sets: by extreme points and by supporting hyperplanes. Furthermore, we now have two ways of showing that an XOR non-local game requires high entanglement: if the representations of the solution algebra are known then we can apply Corollary 2.4; and if we can show that the optimization problem (Φ) has a unique solution of high rank then this solution must be an extreme point of C_{mn} and we can apply Theorem 2.7. Both approaches work for the examples of games requiring high entanglement presented in Section 7.

This raises the question of whether Theorems 2.3 and 2.4 tell us anything new about optimal strategies and entanglement requirements, especially since \mathcal{A} is almost always Clifford. The answer is yes, for interesting reasons. Theorem 2.7 does not give a full description of the optimal strategies for a game G unless (Φ) has a unique solution.

Proposition 2.8. *The solution algebra \mathcal{A} for a game G with no zero rows or columns is Clifford if and only if there is a unique quantum correlation which is optimal for G .*

Proof. For \mathcal{A} to be Clifford means that $X_i X_j + X_j X_i = 2V_{ij} \mathbb{1}$ for all i, j and some fixed $n \times n$ matrix V . In Proposition 3.3, we will give an optimization problem (Γ) with the property that $X_i X_j + X_j X_i = 2V_{ij} \mathbb{1}$ in some representation of \mathcal{A} if and only if V is an optimal solution to (Γ) . When G has no zero rows or columns there is a one-to-one relationship between optimal solutions of (Γ) and optimal quantum correlations. If \mathcal{A} is Clifford then (Γ) will have a unique solution, so there must be a unique optimal quantum correlation.

On the other hand, if there is a unique quantum correlation, then (Γ) will also have a unique solution. By Theorem 2.7, the relation $X_i X_j + X_j X_i = 2V_{ij} \mathbb{1}$ will hold in every representation of \mathcal{A} , with V the unique solution to (Γ) , so this relation holds in \mathcal{A} itself. \square

In addition, the strategy of minimum entanglement is not always Clifford.

Proposition 2.9. *For any $n \geq 1$ there is a XOR non-local game such that any optimal strategy which is Clifford uses at least n ebits more than the strategy of minimum entanglement. For an example, see the game $CL(n)$ described in Section 7.*

In particular, the correlation of a strategy of minimum entanglement does not have to be an extreme point of C_{mn} , and the minimum entanglement required by a game is not a function of the rank of the optimal quantum correlation.

2.5 Stability and approximate strategies

Stability questions arise naturally in applications of Theorem 2.3. For example, suppose we only have an approximation \hat{c}_i to the marginal biases c_i . Are the representations of the algebra deter-

²Tsirelson actually proves this result for a slightly different definition of quantum strategy, but the result still holds for the definition in use here.

mined by the relations

$$X_j^2 = \mathbb{1} \text{ for all } 1 \leq j \leq n, \text{ and } \left(\sum_j G_{ij} X_j \right)^2 = \hat{c}_i^2 \cdot \mathbb{1} \text{ for all } 1 \leq i \leq m.$$

close to representations of the solution algebra \mathcal{A} ? This question is related to the study of stable relations for C^* -algebras, and the broader area of Hyers-Rasiass-Ulam stability in functional analysis and group theory. We use the following definition of stability due to [13]—note that a norm condition $\|B_i\| \leq \|X_i\|_{\mathcal{A}}$ is added to simplify some of the later analysis.

Definition 2.10. *Let \mathcal{A} be a universal C^* -algebra generated by self-adjoint indeterminates X_1, \dots, X_n satisfying relations R_1, \dots, R_k . An ϵ -approximate representation of \mathcal{A} is a collection of bounded self-adjoint operators B_1, \dots, B_n on a Hilbert space H , such that $\|B_i\| \leq \|X_i\|_{\mathcal{A}}$ for all $1 \leq i \leq n$ and $\|R_i(B_1, \dots, B_n)\| \leq \epsilon$ for all $1 \leq i \leq k$.*

A set of relations R_1, \dots, R_k is stable (for finite-dimensional Hilbert spaces) if there is a constant $\delta > 0$ and function $f : [0, \delta) \rightarrow [0, +\infty)$ such that if $0 < \epsilon < \delta$ and (B_1, \dots, B_n) is an ϵ -approximate representation of \mathcal{A} on a finite-dimensional Hilbert space H then there is a representation B'_1, \dots, B'_n of \mathcal{A} on H with $\|B_i - B'_i\| \leq f(\epsilon)$.

A note on norms: we use $\|\cdot\|$ to denote the operator norm induced by the Hilbert space structure. On finite-dimensional Hilbert spaces we can also use the Frobenius norm $\|X\|_F = \sqrt{\text{tr}(X^*X)}$. The two norms are related by the inequality $\|\cdot\| \leq \|\cdot\|_F \leq \sqrt{d} \|\cdot\|$, where d is the dimension.

Proposition 2.11.

- The Clifford algebra C_r is stable with constant $\delta = 1/(250r^2)$ and function $f(\epsilon) = 5r^2\epsilon/2$.
- If \mathcal{A} is strongly Clifford of rank r then there is a constant $\delta > 0$, such that if \mathcal{A} has an ϵ -approximate representation with $\epsilon < \delta$ then there is an exact representation of \mathcal{A} of the same dimension.
- Almost all $m \times n$ games with $m \geq \binom{n}{2}$ are strongly Clifford.

The proof of Proposition 2.11 is given in Section 6. Although we do not make use of the fact, the solution algebra of an $m \times n$ game is stable if it is strongly Clifford of rank n . The author does not know whether or not the relations for the solution algebra are stable in general.

A quantum strategy is said to be ϵ -optimal for a game G if it has success bias within ϵ of $\varepsilon_q(G)$. It is possible to use the solution algebra to study ϵ -optimal strategies as $\epsilon \rightarrow 0$. It will be clear from the proof of Theorem 2.3 that an approximate representation of \mathcal{A} will be the marginal of a near-optimal strategy for G . A weak converse is also true.

Theorem 2.12. *For every $m \times n$ XOR-non-local game with no zero rows or columns there are constants $C, C' > 0$ such that if $0 < \epsilon < C$ and $\{B_j : j \in T\}, \rho$ is the marginal of an ϵ -optimal strategy supported on a Hilbert space H with $\dim H = d < +\infty$, then there is a projection P on H such that $\{PB_jP\}$ is a $(C'd^{3/2}\epsilon^{1/8})$ -approximate representation for the solution algebra of G .*

Specifically, $C \geq \min_j d_j^{16}/(100(m+n))$ (d_j is the j th marginal column bias), and $C' \leq 15(m+n)^{\frac{1}{8}}$. The proof of this theorem is given in Section 5.

For near-optimal strategies we would like to understand the dimensions of the Hilbert spaces supporting ϵ -optimal strategies, for ϵ in a neighbourhood of zero. For this to make sense, we look at

finite-dimensional spaces only. Cleve, Hoyer, Toner, and Watrous have shown that an $m \times n$ XOR non-local game has an ϵ -optimal quantum strategy on a Hilbert space of dimension $(m+n+1)^{O(\epsilon^{-2})}$ [8]. Regev has pointed out that this bound can be made independent of m and n , so that fixing the precision ϵ upper bounds the entanglement cost for all XOR non-local games [10]. Theorem 2.12 gives lower bounds on the dimension of approximate solutions when the solution algebra \mathcal{A} is strongly Clifford.

Corollary 2.13. *Let G be an XOR non-local game with no zero rows or columns, and solution algebra \mathcal{A} which is strongly Clifford of rank r . Then there are constants C and C'' such that if $0 < \epsilon \leq C$ then any ϵ -optimal strategy is supported on a Hilbert space of dimension at least $\min(C''\epsilon^{-1/12}, 2^{\lceil r/2 \rceil})$.*

Proof. Let C, C' to be the constants from Theorem 2.12. Suppose G has an ϵ -optimal solution on a Hilbert space H of dimension d , $0 \leq \epsilon \leq C$. By Theorem 2.12, \mathcal{A} has a $C'd^{3/2}\epsilon^{1/8}$ -approximate representation of dimension d . Take $C'' = (C'/\delta)^{-2/3}$, where δ is the constant from Proposition 2.11. If $d < C''\epsilon^{-1/12}$ then

$$C'd^{3/2}\epsilon^{1/8} < \delta,$$

so Proposition 2.11 implies that \mathcal{C}_r has an exact representation on H , implying that $d \geq 2^{\lceil r/2 \rceil}$. \square

The lower bound in Corollary 2.13 appears to be quite weak. For instance, it is possible to show that dimension increases linearly with $1 - \epsilon$ up to dimension n , for a specific family of games with question sets of size $n(n-1)$ and n (see Proposition 7.2).

3 Marginal biases

Theorem 3.1. *For every $m \times n$ XOR non-local game G there is a collection of non-negative constants $\{c_i : i \in S\}$ such that if a vector strategy $\{u_i\}, \{v_j\}$ for G is ϵ -optimal, $0 \leq \epsilon < 1/(4(m+n))$, then*

$$\left\| \sum_j G_{ij}v_j - c_i u_i \right\| \leq \sqrt{10}(m+n)^{1/4}\epsilon^{1/4}.$$

If G has no zero rows then the constants c_i are non-zero.

When $\epsilon = 0$, $\sum_j G_{ij}v_j = c_i u_i$, so the sums $\sum_j G_{ij}u_i \cdot v_j$ are constant (and equal to c_i) for every optimal vector strategy $\{u_i\}, \{v_j\}$. There are also *marginal column biases* d_j playing the same role for the columns. The marginal column biases are non-zero when G has no zero columns.

Corollary 3.2. *Let G be an XOR non-local game with no zero rows. A vector strategy $\{u_i\}, \{v_j\}$ for G is optimal if and only if*

$$\sum_j G_{ij}v_j = c_i u_i \text{ for all } 1 \leq i \leq m,$$

where c_i is the i th marginal row bias of G .

Alternatively, we can characterize optimal vector strategies without any reference to the vectors u_i at all.

Corollary 3.3. *Let $\{v_j\}$ be a collection of n unit vectors. Then there are unit vectors $\{u_i\}$ such that the quantum correlation $\{u_i \cdot v_j\}$ is an optimal solution to G if and only if the positive semidefinite matrix $\{v_i \cdot v_j\}$ is an optimal solution to*

$$(\Gamma) \quad \max \sum_i \sqrt{\sum_{j,k} G_{ij} G_{ik} V_{jk}} : V \text{ is an } n \times n \text{ positive semidefinite matrix with } V_{ii} = 1.$$

If G has no zero rows or columns there is a one-to-one relationship between solutions to (Γ) and optimal quantum correlations for G . If V is an optimal solution then the i th marginal row bias is given by $c_i^2 = \sum_{j,k} G_{ij} G_{ik} V_{jk}$.

To prove Theorem 3.1, consider the semidefinite programming (SDP) formulation for optimal value of G given in [9]. Specifically, let

$$B = \begin{pmatrix} 0 & \frac{1}{2}G \\ \frac{1}{2}G^T & 0 \end{pmatrix}.$$

Then $\varepsilon_q(G)$ is equal to

$$(P) \quad \max \operatorname{tr}(BX) : X_{ii} = 1 \text{ for } 1 \leq i \leq m+n, \text{ and } X \succeq 0,$$

where \succeq denotes the partial order induced by the semidefinite cone. An overview of semidefinite programming techniques can be found in [3]. There is a direct correspondence between vector strategies and feasible solutions of (P) : if $\{u_i\}, \{v_j\}$ is a vector strategy, U is the matrix with columns given by the vectors u_i , and V is the matrix with columns given by the vectors v_j , then the corresponding feasible solution to (P) is the matrix

$$X = \begin{pmatrix} U^T \\ V^T \end{pmatrix} \cdot \begin{pmatrix} U & V \end{pmatrix} = \begin{pmatrix} U^T U & U^T V \\ V^T U & V^T V \end{pmatrix}, \quad (1)$$

and this construction can be run in reverse as well. The success bias can also be calculated using the dual formulation:

$$(D) \quad \min \frac{1}{2} \sum_{1 \leq i \leq m} c_i + \frac{1}{2} \sum_{1 \leq i \leq n} d_i : \\ \frac{1}{2} \begin{pmatrix} \Delta(c) & 0 \\ 0 & \Delta(d) \end{pmatrix} \succeq B.$$

Here $\Delta(c)$ denotes the diagonal matrix with the entries of c on the diagonal.

Proof of Theorem 3.1. Let $c_1, \dots, c_m, d_1, \dots, d_n$ be optimal for (D) , and let

$$S = \frac{1}{2} \begin{pmatrix} \Delta(c) & 0 \\ 0 & \Delta(d) \end{pmatrix} - B.$$

Let X be the positive semidefinite matrix corresponding to a (not necessarily optimal) vector strategy $\{u_i\}, \{v_j\}$. Then

$$(SX)_{ii} = \begin{cases} \frac{1}{2}c_i - \frac{1}{2} \sum_j G_{ij} u_i \cdot v_j & 1 \leq i \leq m \\ \frac{1}{2}d_{i-m} - \frac{1}{2} \sum_j G_{j(i-m)} u_j \cdot v_{i-m} & m+1 \leq i \leq m+n \end{cases},$$

and in particular

$$\text{tr}(SX) = \frac{1}{2} \sum_{1 \leq i \leq m} c_i + \frac{1}{2} \sum_{1 \leq j \leq n} d_j - \text{tr}(BX) = \varepsilon_q(G) - \text{tr}(BX),$$

where the last equality holds by strong duality. The vector strategy in question is ϵ -optimal if and only if X is ϵ -optimal for (P) , meaning that $\varepsilon_q(G) - \text{tr}(BX) \leq \epsilon$. The following inequality holds for every $1 \leq i \leq m$ and every ϵ -optimal vector strategy:

$$\begin{aligned} \left| c_i - \sum_j G_{ij} u_i \cdot v_j \right| &= 2|(SX)_{ii}| \leq 2\|SX\|_F \leq 2\|X^{1/2}\|_F \|S^{1/2}\|_F \|X^{1/2}S^{1/2}\|_F \\ &= 2\sqrt{\text{tr}(X)}\sqrt{\text{tr}(S)}\sqrt{\text{tr}(XS)} \leq 2\sqrt{m+n}\sqrt{\varepsilon_q(G)}\sqrt{\epsilon}, \end{aligned}$$

where $\|\cdot\|_F$ is the Frobenius norm. Let $\delta = 2\sqrt{m+n}\sqrt{\varepsilon_q(G)}\sqrt{\epsilon}$. Then

$$\left\| c_i u_i - \sum_j G_{ij} v_j \right\|^2 = c_i^2 - 2c_i \sum_j G_{ij} u_i \cdot v_j + \left\| \sum_j G_{ij} v_j \right\|^2.$$

We know that $\sum_j G_{ij} u_i \cdot v_j > c_i - \delta$. To bound $\left\| \sum_j G_{ij} v_j \right\|$, let u'_i be the vector $\sum_j G_{ij} v_j$, renormalized to a unit vector. Then

$$\sum_j G_{ij} u'_i \cdot v_j = \left\| \sum_j G_{ij} v_j \right\| \geq \sum_j G_{ij} u_i \cdot v_j,$$

so $\{u'_i\}, \{v_j\}$ is also an ϵ -optimal strategy for G . Hence $\left\| \sum_j G_{ij} v_j \right\| = \sum_j G_{ij} u'_i \cdot v_j \leq c_i + \delta$. We can conclude that

$$\left\| c_i u_i - \sum_j G_{ij} v_j \right\|^2 \leq c_i^2 - 2c_i(c_i - \delta) + (c_i + \delta)^2 = 4c_i\delta + \delta^2,$$

and this last expression is at most $5\varepsilon_q(G)^{1/2}\delta$ if $\epsilon \leq 1/(4(m+n))$. Since $\epsilon_q(G) \leq 1$, this shows that the constants c_i satisfy the statement of Theorem 3.1. Uniqueness follows from the fact that $c_i = \sum_j G_{ij} u_i \cdot v_j$ for every optimal vector strategy $\{u_i\}, \{v_j\}$. Finally S is positive semidefinite, so if the i th row of G is non-zero then c_i must be strictly positive. \square

4 Quantum strategies: the exact case

We begin with the proof of Theorem 2.3. For this we introduce some notation, which we will keep fixed for this and also the next section. Suppose $|\psi\rangle$ is an element of $H_1 \otimes H_2$. If we pick a basis $|i\rangle$ for H_1 then we can write $|\psi\rangle = \sum |i\rangle \lambda |i\rangle$, where $\lambda : H_1 \rightarrow H_2$ is a linear transformation. With this notation, the Schmidt coefficients for $|\psi\rangle$ are the eigenvalues of λ , and the partial trace of $|\psi\rangle$ with respect to H_1 is $\rho = \lambda \lambda^*$, a density operator on H_2 . Having picked a basis on H_1 , we can write any element of $\mathcal{B}(H_1)$ as a matrix. If $A \in \mathcal{B}(H_1)$, let \bar{A} denote the linear transformation constructed by taking the entry-wise complex conjugate of the matrix of A . Note that the operation $A \mapsto \bar{A}$ depends on the choice of basis for H_1 .

Lemma 4.1. *Let A and B be Hermitian operators on Hilbert spaces H_1 and H_2 respectively. Then $\|(A \otimes \mathbb{1} - \mathbb{1} \otimes B)|\psi\rangle\| = \|\lambda\bar{A} - B\lambda\|_F$. Consequently, if $\|(A \otimes \mathbb{1} - \mathbb{1} \otimes B)|\psi\rangle\| \leq \epsilon$, then $\|\rho B - B\rho\|_F \leq 2\epsilon$, or in other words ρ approximately commutes with B .*

Proof. For the first identity, observe that $(A \otimes \mathbb{1})|\psi\rangle = \sum_i |i\rangle \lambda \bar{A} |i\rangle$, while $(\mathbb{1} \otimes B)|\psi\rangle = \sum_i |i\rangle B \lambda |i\rangle$. So $\|(A \otimes \mathbb{1} - \mathbb{1} \otimes B)|\psi\rangle\|^2 = \|\lambda\bar{A} - B\lambda\|_F^2$.

If $\|(A \otimes \mathbb{1} - \mathbb{1} \otimes B)|\psi\rangle\| \leq \epsilon$ then $\|B\rho - \lambda\bar{A}\lambda^*\|_F \leq \|B\lambda - \lambda\bar{A}\|_F \|\lambda^*\|_F \leq \epsilon$. But $\lambda\bar{A}\lambda^*$ is Hermitian, while $(B\rho)^* = \rho B$. \square

Now we should say something about what it means for an operator strategy to be non-degenerate. Suppose that $\{A_i\}, \{B_j\}, |\psi\rangle$ is a quantum strategy, and let \mathcal{B}_1 and \mathcal{B}_2 denote the C^* -algebras generated by A_1, \dots, A_m and B_1, \dots, B_n respectively. Let cl denote the closure of a set in a Hilbert space with respect to the norm topology.

Lemma 4.2. *The strategy in question is non-degenerate if and only if $\text{cl } \mathcal{B}_2 \lambda H_1 = H_2$ and $\text{cl } \bar{\mathcal{B}}_1 \lambda^* H_2 = H_1$.*

Proof. Let Q be a projection onto a closed subspace W of H_2 . By Lemma 4.1, $(I \otimes Q)|\psi\rangle = |\psi\rangle$ if and only if $Q\lambda = \lambda$, and this occurs if and only if the image of λ is contained in W . Also Q will commute with \mathcal{B}_2 if and only if W is an invariant subspace for \mathcal{B}_2 . Thus there is a projection Q commuting with \mathcal{B}_2 such that $(I \otimes Q)|\psi\rangle = |\psi\rangle$ if and only if $\text{cl } \mathcal{B}_2 \lambda H_1$ is a strict subset of H_2 .

There is a projection P such that $(P \otimes I)|\psi\rangle = |\psi\rangle$ if and only if $\lambda\bar{P} = \lambda$, and this is equivalent to the condition that $\bar{P}\lambda^* = \lambda^*$ (note that \bar{P} is also a projection). P will commute with \mathcal{B}_1 if and only if \bar{P} commutes with $\bar{\mathcal{B}}_1$. Thus we arrive at the requirement that $\text{cl } \bar{\mathcal{B}}_1 \lambda^* H_2 = H_1$. \square

From this lemma it is easy to see that every strategy projects down to a unique non-degenerate strategy. One more special case is important. The kernel of λ is the orthogonal complement of the image of λ^* , so $\text{cl } \lambda H_1 = \text{cl } \rho H_2$. If ρ commutes with \mathcal{B}_2 then ρH_2 is an invariant subspace of \mathcal{B}_2 , so $\text{cl } \mathcal{B}_2 \lambda H_1 = H_2$ if and only if $\text{cl } \rho H_2 = H_2$.

Proof of Theorem 2.3. Suppose we are given an optimal non-degenerate strategy $\{A_i\}, \{B_j\}, |\psi\rangle$ for G , acting on Hilbert spaces H_1 and H_2 . We fix a basis for H_1 , and use the notation introduced above for $|\psi\rangle$. Define $u_i = (A_i \otimes I)|\psi\rangle$ and $v_j = (I \otimes B_j)|\psi\rangle$. The vectors u_1, \dots, u_m and v_1, \dots, v_n form an optimal vector strategy for G . Let c_i denote the i th marginal row bias for G , and d_j the j th marginal column bias. By Corollary 3.2, we know that

$$d_j(I \otimes B_j)|\psi\rangle = \sum_i G_{ij}(A_i \otimes I)|\psi\rangle,$$

where each d_j is non-zero. It follows from Lemma 4.1 that ρ commutes with all B_j . As previously mentioned, Lemma 4.2 now implies that the closure of the image of ρ is equal to H_2 .

Now we also know that

$$c_i(A_i \otimes I)|\psi\rangle = \sum_j G_{ij}(I \otimes B_j)|\psi\rangle,$$

and consequently by Lemma 4.1 $\sum_j G_{ij} B_j \lambda = c_i \lambda \bar{A}_i$. We apply the map $S \mapsto SS^*$ to both sides to get

$$\left(\sum_j G_{ij} B_j \right)^2 \rho = \left(\sum_j G_{ij} B_j \right) \rho \left(\sum_j G_{ij} B_j \right) = c_i^2 \lambda \bar{A}_i^2 \lambda^* = c_i^2 \rho.$$

Since the closure of the image of ρ is equal to H_2 , we find that $\left(\sum_j G_{ij}B_j\right)^2 = c_i^2 I$. Thus the operators $\{B_j\}, \rho$ describe a cyclic representation of \mathcal{A} , and the density operator ρ commutes with \mathcal{A} .

Conversely, suppose we are given a cyclic representation $\{B_j\}, \rho$ of \mathcal{A} , acting on a Hilbert space H , such that ρ commutes with all B_i and $\text{cl } \mathcal{A}\rho H = H$. We can conclude that $\text{cl } \rho H = H$, so if λ is the square root of ρ , then $\rho H \subset \lambda H$, and hence $\text{cl } \lambda H = H$. Now set $H_1 = H_2 = H$ and $|\psi\rangle = \sum_i |i\rangle(\lambda|i\rangle)$. Define $\bar{A}_i = \frac{1}{c_i} \sum_j G_{ij}B_j$, so that A_i is a self-adjoint operator squaring to the identity. From Lemma 4.1 we can conclude that

$$c_i(A_i \otimes I)|\psi\rangle = \sum_j G_{ij}(I \otimes B_j)|\psi\rangle,$$

and hence

$$\sum_{i,j} G_{ij} \langle \psi | A_i \otimes B_j | \psi \rangle = \sum_i c_i$$

is the optimal success bias for G . Thus $\{A_i\}, \{B_j\}, |\psi\rangle$ is an optimal quantum strategy for G . It is non-degenerate because $\text{cl } \lambda H = H$ and $\lambda^* = \lambda$. \square

Now we can give the proof of Corollary 2.4.

Proof of Corollary 2.4. Let $\{B_j\}, \rho$ be a cyclic representation of \mathcal{A} on a Hilbert space H such that ρ commutes with all B_j . Let α_i be the eigenvectors of ρ . By the spectral theorem for self-adjoint compact operators (see [21] for the statement of this theorem), H decomposes into a direct sum of finite-dimensional eigenspaces E_{α_i} of ρ . If $v \in E_{\alpha_i}$ is an eigenvector, then $\rho B_j v = B_j \rho v = \alpha_i B_j v$, and hence \mathcal{A} preserves the eigenspaces of ρ . Because each E_{α_i} is finite-dimensional, each decomposes further into a sum of irreducible representations of \mathcal{A} . Thus we can conclude that H decomposes into a direct sum of finite-dimensional irreducible representations H_k .

Now we can write ρ as a sum of density operators which have orthogonal support. Namely, let $\rho_k = I/\text{tr}(I)$, where I denotes the identity on H_k . Then $\rho = \sum p_k \rho_k$, where the p_k 's are positive scalars summing to one. By Theorem 11.8 of [14], $S(\rho) = \sum p_k S(\rho_k) + H(p_k)$, where S denotes the von Neumann entropy and H denotes the Shannon entropy. $H(p_k) > 0$ unless there is exactly one invariant subspace. Hence the minimum entanglement will be attained only by an irreducible representation. The entanglement used by an irreducible representation is $S(\rho_k) = \log_2(\dim H_k)$. \square

5 Quantum strategies: the approximate case

The goal of this section is to prove Theorem 2.12. We rely on the following lemma, which is a concatenation of lemmas concerning eigenvalue gaps due to Babai and Friedl.

Lemma 5.1 (Babai and Friedl, 1991). *Let ρ be a density matrix on the finite dimensional Hilbert space \mathbb{C}^d such that $\tau := \|\rho - \mathbb{1}/d\| > 0$. Then there is an orthogonal decomposition $\mathbb{C}^d = W_1 \oplus W_2$ such that*

- W_1 and W_2 are non-trivial invariant subspaces of ρ ,
- if α_1 and α_2 are eigenvalues of ρ with eigenvectors in W_1 and W_2 respectively, then $\alpha_1 \geq \alpha_2 + \tau/d$, and

- if ρ approximately commutes with another matrix S , meaning that $\|\rho S - S\rho\|_F \leq \epsilon$, then

$$\|P_1 S P_1 + P_2 S P_2 - S\|_F \leq \frac{\epsilon d}{\tau},$$

where P_i is the orthogonal projection onto W_i .

Proof. Write the eigenvalues of ρ in descending order as $\lambda_1 \geq \dots \geq \lambda_d$. Since $\sum \lambda_i = 1$, we must have $\lambda_1 \geq 1/d \geq \lambda_n$. Let $\Delta = \max |\lambda_i - \lambda_{i+1}|$ be the largest eigenvalue gap. Then

$$\tau = \max |\lambda_i - 1/d| \leq \lambda_1 - \lambda_d \leq d\Delta,$$

so $\Delta \geq \tau/d$. Then apply Lemma 2.11 and Remark 2.7 of [1]. \square

Proof of Theorem 2.12. We want to find a projection P on H_2 such that $\|P - (PB_j P)^2\|$ and $\|c_i^2 P - (\sum_l G_{il} P B_l P)^2\|$ are bounded above by $Cd^{5/3}\epsilon^{1/12}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We start by bounding $\|c_i^2 P - (\sum_l G_{il} P B_l P)^2\|$. Let $C' = \sqrt{10}(m+n)^{1/4}$.

Let $|\psi\rangle = \sum |i\rangle \lambda |i\rangle$ and $\rho = \lambda \lambda^*$ be the partial trace of $|\psi\rangle$ with respect to H_1 . If $\{A_i\}, \{B_j\}$, $|\psi\rangle$ is ϵ -optimal, then by Theorem 3.1, $\left\| \left(c_i A_i \otimes \mathbb{1} - \sum_j G_{ij} \mathbb{1} \otimes B_j \right) |\psi\rangle \right\| \leq C' \epsilon^{1/4}$. It follows from Lemma 4.1 that $\left\| c_i \lambda \bar{A} - \sum_j G_{ij} B_j \lambda \right\|_F \leq C' \epsilon^{1/4}$. It follows that $\left\| c_i^2 \rho - \sum_j G_{ij} B_j \rho \sum_j G_{ij} B_j \right\| \leq 3C' \epsilon^{1/4}$. But Lemma 4.1 also tells us that $\left\| \rho \sum_j G_{ij} B_j - \sum_j G_{ij} B_j \rho \right\|_F \leq 2C' \epsilon^{1/4}$ for all j , so combining this with the last inequality, we get

$$\left\| \left(c_i^2 - \left(\sum_j G_{ij} B_j \right)^2 \right) \rho \right\| \leq 5C' \epsilon^{1/4}. \quad (2)$$

Inequality (2) is potentially much weaker than the inequality we want, since ρ might have arbitrarily small eigenvalues. Lemma 5.1 suggests two ways to strengthen this inequality. The first is to replace ρ with the maximally mixed density matrix: let $\tau = \|\rho - \mathbb{1}/d\|$ and observe that

$$\begin{aligned} \left\| c_i^2 - \left(\sum_j G_{ij} B_j \right)^2 \right\| &\leq d \left\| \left(c_i^2 - \left(\sum_j G_{ij} B_j \right)^2 \right) \rho \right\| \\ &\quad + d \left\| c_i^2 - \left(\sum_j G_{ij} B_j \right)^2 \right\| \|\rho - \mathbb{1}/d\| \leq 5C' d \epsilon^{1/4} + 2d\tau. \end{aligned}$$

If $\tau = 0$ then we can take $P = \mathbb{1}$ and we will be done. If $\tau > 0$ then we can apply Lemma 5.1 to decompose H_2 as $W_1 \oplus W_2$. Let P_i denote the orthogonal projection onto W_i . Let $B'_j = P_1 B_j P_1 + P_2 B_j P_2$ and let $\rho_i = P_i \rho P_i$. Because $\sum_j G_{ij} B_j$ approximately commutes with ρ , we get $\left\| \sum_j G_{ij} B_j - \sum_j G_{ij} B'_j \right\| \leq 2C' d \epsilon^{1/4} / \tau$. If we let $D_{kl} = \sum_j G_{ij} P_k B_j P_l$, then

$$\sum_j G_{ij} B_j - \sum_j G_{ij} B'_j = \begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix},$$

so $\|D_{12}\|, \|D_{21}\| \leq 2C' d \epsilon^{1/4} / \tau$, and

$$\left\| (c_i^2 P_1 - D_{11}^2 - D_{12} D_{21}) \rho_1 \right\| \leq 5C' \epsilon^{1/4}.$$

But $\|D_{12}D_{21}\| \leq 4C'd^2\epsilon^{1/2}/\tau^2$, so

$$\|(c_i^2 - D_{11}^2)\rho_1\| \leq 5C'\epsilon^{1/4} + 4(C')^2d^2\epsilon^{1/2}/\tau^2.$$

Since ρ_1 has minimum eigenvalue at least τ/d , we can conclude that

$$\begin{aligned} \left\| c_i^2 P_1 - \left(\sum_j G_{ij} P_1 B_j P_1 \right)^2 \right\| &\leq \left\| \left(c_i^2 P_1 - \left(\sum_j G_{ij} P_1 B_j P_1 \right)^2 \right) \rho_1 \right\| \|(\rho_1)^{-1}\| \\ &\leq \frac{5C'd\epsilon^{1/4}}{\tau} + \frac{4(C')^2d^3\epsilon^{1/2}}{\tau^3}. \end{aligned}$$

This leaves us to compare the efficacy of two choices for the projection P required by the proposition. If we choose $P = \mathbb{1}$, then we have an upper bound on $\left\| c_i^2 P - \left(\sum_j G_{ij} P B_j P \right)^2 \right\|$ given by $f_1(\tau) = 5C'd\epsilon^{1/4} + 2d\tau$, while if we choose $P = P_1$ we have an upper bound given by $f_2(\tau) = 5C'd\epsilon^{1/4}/\tau + 4(C')^2d^3\epsilon^{1/2}/\tau^3$. Let $\tau_0 = \frac{3}{2}(C')^{1/2}d^{1/2}\epsilon^{1/8}$, and observe that $f_1(\tau_0)$ and $f_2(\tau_0)$ are both bounded above by $8(C')^{1/2}d^{3/2}\epsilon^{1/8}$. Since f_1 is increasing, if $\tau \leq \tau_0$ then we can take $P = \mathbb{1}$ and get the inequality required by the proposition. If $\tau > \tau_0$, then f_2 is decreasing on the interval $(0, \infty)$, so we can take $P = P_1$ and get the inequality required by the proposition.

So far we have ignored the inequality we are supposed to get on $\|P - (PB_jP)^2\|$. If we take $P = \mathbb{1}$, then $P = (PB_jP)^2$. So suppose $\tau > \tau_0$ and we take $P = P_1$. Now by Theorem 3.1 applied to the columns, $\|(d_j \mathbb{1} \otimes B_j - \sum_i G_{ij} A_i \otimes \mathbb{1})\psi\| \leq C'\epsilon^{1/4}$. Hence $\|B_j\rho - \rho B_j\|_F \leq 2C'\epsilon^{1/4}/d_j$. As before, Lemma 5.1 tells us that $\|P_1 B_j P_1\|, \|P_2 B_j P_1\| \leq 2C'd\epsilon^{1/4}/(d_j\tau)$, so

$$\left\| P_1 - (P_1 B_j P_1)^2 \right\| \leq \frac{4(C')^2d^2\epsilon^{1/2}}{d_j^2\tau^2}.$$

Let $f_3(\tau) = 4(C')^2d^2\epsilon^{1/2}/(d_j^2\tau^2)$. Then f_3 is decreasing and

$$f_3(\tau_0) = \frac{8C'd\epsilon^{1/4}}{9d_j^2} \leq (C')^{1/2}d\epsilon^{1/8},$$

where the last inequality comes from the hypothesis that $\epsilon \leq d_j^{16}/(C')^4$. So we get $f_3(\tau) \leq 8(C')^{1/2}d^{3/2}\epsilon^{1/8}$ as long as $\tau > \tau_0$. \square

6 Stable relations for the Clifford algebra

We use the following stability result for amenable groups due to Kazhdan to prove that the relations defining the Clifford algebra are stable.

Definition 6.1. *An ϵ -representation of a topological group G is a continuous map ϕ from G to the group of unitary transformations on a Hilbert space, such that*

$$\|\phi(gg') - \phi(g)\phi(g')\| \leq \epsilon \text{ for all } g, g' \in G.$$

Here $\|\cdot\|$ is the operator norm.

Theorem 6.2 (Kazhdan, Theorem 1 of [12]). *Let G be an amenable group and $\phi : G \rightarrow U$ an ϵ -representation of G , for some $0 < \epsilon < 1/100$. Then there is a representation $\pi : G \rightarrow U$ such that $\|\phi(g) - \pi(g)\| \leq \epsilon$ for all $g \in G$.*

Let G_n be the multiplicative subgroup generated by the elements Y_1, \dots, Y_n in the Clifford algebra \mathcal{C}_n . As a set, G_n is equal to

$$\{J^{a_0} Y_0^{a_1} \dots Y_n^{a_n} : a_0, \dots, a_n \in \{0, 1\}\}$$

where we use J to denote $-\mathbb{1}$ for clarity. G_n is a finite group of order 2^{n+1} , and hence is an amenable group with the discrete topology.

Lemma 6.3. *Suppose B_1, \dots, B_n is an ϵ -approximate representation of \mathcal{C}_n on the finite-dimensional Hilbert space \mathbb{C}^d , where $0 < \epsilon < 1/(250n^2)$. Then there is a representation B'_1, \dots, B'_n of \mathcal{C}_n on \mathbb{C}^d such that $\|B_i - B'_i\| \leq 5n^2\epsilon/2$.*

Proof. Since B_i is self-adjoint and $\|B_i^2 - \mathbb{1}\| \leq \epsilon$, every eigenvalue of B_i is within ϵ of either $+1$ or -1 . Hence there is a self-adjoint unitary matrix \hat{B}_i such that $\|B_i - \hat{B}_i\| \leq \epsilon$. Since $\|B_i B_j + B_j B_i\| \leq \epsilon$, it follows that $\|\hat{B}_i \hat{B}_j + \hat{B}_j \hat{B}_i\| \leq 5\epsilon$. Suppose $J^{a_0} Y_1^{a_1} \dots Y_n^{a_n} \cdot J^{b_0} Y_1^{b_1} \dots Y_n^{b_n} = J^{c_0} Y_1^{c_1} \dots Y_n^{c_n}$. This means that the product $J^{a_0} Y_1^{a_1} \dots Y_n^{a_n} \cdot J^{b_0} Y_1^{b_1} \dots Y_n^{b_n}$ can be transformed to $J^{c_0} Y_1^{c_1} \dots Y_n^{c_n}$ using the relations $Y_i Y_j = -Y_j Y_i$ at most $n(n-1)/2$ times. So the map

$$\psi : J^{a_0} Y_1^{a_1} \dots Y_n^{a_n} \mapsto (-1)^{a_0} \hat{B}_1^{a_1} \dots \hat{B}_n^{a_n}$$

gives a unitary ϵ' -representation of G_n , where $\epsilon' = 5\binom{n}{2}\epsilon$. By Kazhdan's theorem, there is a representation ϕ of G_n with $\|\phi - \psi\| \leq \epsilon'$ and $\|\phi(Y_i) - B_i\| \leq \epsilon' + \epsilon \leq 5n^2\epsilon/2$. Let $B'_i = \phi(Y_i)$. Now $B'_i B'_j = \phi(J) B'_j B'_i$, so we will be done if we can show that $\phi(J) = -\mathbb{1}$. But $\psi(J) = -\mathbb{1}$, so $\|\psi(J) + \mathbb{1}\| \leq \epsilon' < 1$. Since $\psi(J)$ is unitary and squares to $\mathbb{1}$, it follows that $\psi(J) = -\mathbb{1}$. \square

Proof of Proposition 2.11. Let E^{kl} denote the $n \times n$ matrix with ones in the kl th and lk th positions, and zeroes everywhere else. Map every $m \times n$ XOR non-local game G to an m -tuple of rank one $n \times n$ matrices G^1, \dots, G^m , where $G_{kl}^i = G_{ik} G_{il}$. If the i th row of G is equal to the vector with ones in the l th and k th positions and zeroes elsewhere, then G^i will be equal to $E^{kk} + E^{ll} + E^{kl}$. Thus whenever $m \geq \binom{n}{2}$, there is at least one XOR non-local game such that the matrices $G^1, \dots, G^m, E^{11}, \dots, E^{nn}$ span the space of $n \times n$ symmetric matrices. We can think of the space of $m \times n$ XOR non-local games either as \mathbb{R}^{mn} (without normalization) or as the $mn - 1$ -sphere in \mathbb{R}^{mn} (with normalization). In either case, the subset of XOR non-local games for which the matrices $G^1, \dots, G^m, E^{11}, \dots, E^{nn}$ are linearly dependent is an algebraic set. Hence for almost all $m \times n$ games the matrices $G^1, \dots, G^m, E^{11}, \dots, E^{nn}$ span the space of symmetric matrices. Assume that G is a game for which this is true.

Let M be the $n \times n$ matrix with coefficients $M_{ij} = X_i X_j$ in \mathcal{A} . The defining relations for \mathcal{A} can be rewritten in this notation as

$$E^{jj} \cdot M = \mathbb{1}_{\mathcal{A}} \text{ and } G^i \cdot M = c_i^2 \cdot \mathbb{1}_{\mathcal{A}},$$

where $X \cdot Y$ denotes the coordinate-wise \mathcal{A} -valued scalar product. For every $1 \leq k, l \leq n$ there are scalars s_j and t_i such that

$$\sum_j s_j E^{jj} + \sum_i t_i G^i = E^{kl}.$$

Let $V_{kl} = \sum_i t_i c_i^2 + \sum_j s_j$. Then

$$X_k X_l + X_l X_k - V_{kl} \mathbb{1}_{\mathcal{A}} = E^{kl} \cdot M - V_{kl} \tag{3}$$

$$= \sum_j s_j (E^{jj} \cdot M - \mathbb{1}_{\mathcal{A}}) + \sum_i t_i (G^i \cdot M - c_i^2 \mathbb{1}_{\mathcal{A}}). \tag{4}$$

So for almost all $m \times n$ games the solution algebra is strongly Clifford.

Now assume that \mathcal{A} is strongly Clifford. There are self-adjoint elements $Y_1, \dots, Y_r \in \mathcal{A}$ satisfying the usual Clifford relations, where $r = \text{rank } V$. Each Y_i is a linear combination $\sum y_{ij} X_j$ of the X_j 's, and $2\delta_{ik} = Y_i Y_k + Y_k Y_i = \sum_{j,l} y_{ij} y_{kl} (X_j X_l + X_l X_j) = \sum_{j,l} y_{ij} y_{kl} V_{jl}$. Suppose (B_1, \dots, B_n) is an ϵ -approximate representation of \mathcal{A} , and let $\tilde{Y}_i = \sum y_{ij} B_j$. Then

$$\tilde{Y}_i \tilde{Y}_k + \tilde{Y}_k \tilde{Y}_i - 2\delta_{ik} = \sum_{j,l} y_{ij} y_{kl} (B_j B_l + B_l B_j - V_{jl} \mathbb{1}).$$

The relations $X_j X_l + X_l X_j - V_{jl} \mathbb{1}$ are in turn given by equation (3) for some choice of constants s_j and t_i . It follows that $\left\| \tilde{Y}_i \tilde{Y}_k + \tilde{Y}_k \tilde{Y}_i - 2\delta_{ik} \right\| \leq D' \epsilon$ for some (possibly large) constant D' . In particular $\left\| \tilde{Y}_i \right\| \leq 1 + D' \epsilon / 4$. If we assume $D' \epsilon \leq 1$ then we can truncate eigenvalues of \tilde{Y}_i to 1 to get a $5D' \epsilon / 2$ -approximate representation of C_r . Take $\delta = 1/(625D'r^2)$. If $\epsilon < \delta$ then $\epsilon D' < 1$ and $5D' \epsilon / 2 < 1/(250r^2)$, so we can apply the above lemma to get an exact representation. \square

7 Examples

7.1 XOR non-local games constructed from a graph

There is a general method which allows us to construct many examples of games requiring high entanglement. Start with a graph G with v vertices and e edges. We construct a matrix A which has two rows for each edge of G , and columns indexed by the vertices. If ij is an edge in G with $i < j$, then the first row corresponding to ij will contain a 1 in the i th column, a -1 in the j th column, and zeroes everywhere else. The other row will contain a 1 in both the i th and the j th column, with zeroes everywhere else. Let A_G be the XOR non-local game with game matrix $A/4e$.

For example, if K_2 is the complete graph on 2 vertices, and G is the graph with 3 vertices and edge set $\{12, 13\}$, then

$$A_{K_2} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } A_G = \frac{1}{8} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

Proposition 7.1. *Let A_G be the $2e \times v$ game constructed from the graph G according to the above prescription. Then the optimal solutions to the associated convex programming problem (Γ) of Proposition 3.3 are the semidefinite matrices V satisfying the conditions*

$$V_{ii} = 1 \text{ for all } 1 \leq i \leq v \text{ and } V_{ij} = 0 \text{ for each edge } ij \text{ of } G.$$

The optimal quantum bias for A_G will be $1/2$ and the optimal classical bias will be $1/\sqrt{2}$. The solution algebra for A_G will be the C^ -algebra generated by X_1, \dots, X_v satisfying the relations*

$$X_i^2 = \mathbb{1} \text{ for all } 1 \leq i \leq v \text{ and } X_i X_j = -X_j X_i \text{ for each edge } ij \text{ of } G.$$

Proof of Proposition 7.1. For convenience, let $f(t)$ denote the function $\sqrt{1+t} + \sqrt{1-t}$. The optimization problem (Γ) has objective function

$$\sum_{ij \in E(G)} \frac{1}{2\sqrt{2}e} f(V_{ij}),$$

using the fact that $V_{ii} = 1$ at any feasible point. Now $f(0) = 2$, and this is the unique maximum of $f(t)$ on the interval $[-1, 1]$. If V is feasible for (Γ) then $V_{ij} \in [-1, 1]$, so (Γ) has optimum value $1/\sqrt{2}$, and the optimal solutions are the feasible solutions V with $V_{ij} = 0$ if $ij \in E(G)$.

On the other hand, if V corresponds to a classical solution, then $V_{ij} = \pm 1$, and $f(\pm 1) = \sqrt{2}$.

If V is optimal for (Γ) then the squares of the two row biases corresponding to edge ij are

$$c = \frac{\sqrt{(V_{ij}^2 \pm 2V_{ij}^2 + V_{jj}^2)}}{4e} = \frac{1}{2\sqrt{2}e}.$$

Given that $X_i^2 = X_j^2 = \mathbb{1}$, the relations

$$\left(\frac{X_i}{4e} \pm \frac{X_j}{4e}\right)^2 = c^2 \mathbb{1}$$

are equivalent to the single relation $X_i X_j = -X_j X_i$. □

The game described by A_{K_2} is commonly known as the CHSH game. As an obvious generalization of the CHSH game, we define $\text{CHSH}(n)$ to be the XOR game with game matrix A_{K_n} . The associated solution algebra is strongly Clifford of rank n , and every irreducible representation of the solution algebra has dimension $2^{\lfloor n/2 \rfloor}$. Thus Corollaries 2.4 and 2.13 give lower bounds on the entanglement required for optimal and near-optimal strategies. For $\text{CHSH}(n)$ we also can derive a lower bound on the rank of near-optimal vector strategies.

Proposition 7.2. *Let $\epsilon > 0$. Every ϵ -optimal vector strategy for $\text{CHSH}(n)$ is supported on a Hilbert space of dimension at least $n - 8\sqrt{2}n(n-1)\epsilon$.*

Proof. For the first part of the Proposition, let $\{u_{ijl}\}_{l \in \{0,1\}}$, $\{v_j\}$ be an ϵ -optimal vector strategy. Let V be the matrix with $V_{ij} = v_i \cdot v_j$. Observe that

$$\sum_{1 \leq i < j \leq n} \frac{1}{\sqrt{2}n(n-1)} f(V_{ij}) \geq \sum_{1 \leq i < j \leq n} \frac{1}{2n(n-1)} u_{ij0}(v_i - v_j) + \frac{1}{2n(n-1)} u_{ij1}(v_i + v_j) \geq \frac{1}{\sqrt{2}} - \epsilon,$$

where $f(t) = \sqrt{1+t} + \sqrt{1-t}$ again. Using the Taylor series expansion, we see that $f(t) \leq 2 - t^2/4$ for all $t \in [-1, 1]$. Consequently

$$\sum_{1 \leq i < j \leq n} \frac{1}{\sqrt{2}n(n-1)} f(V_{ij}) \leq \frac{1}{\sqrt{2}} - \frac{\|V - I\|_F^2}{8\sqrt{2}n(n-1)}.$$

Now $\dim \text{span}_{\mathbb{R}}\{v_j\} = \text{rank } V = n - z$, where z is the number of eigenvalues of $V - I$ equal to -1 . But $\|V - I\|_F^2 \geq z$, implying that

$$n - z \geq n - \|V - I\|_F^2 \geq n - 8\sqrt{2}n(n-1)\epsilon.$$

□

7.2 A game for which Tsirelson's bound is tight

Because $\text{CHSH}(n)$ is an $n(n-1) \times n$ game, $\text{CHSH}(n)$ does not quite meet Tsirelson's upper bound on entanglement. To give a game which does meet Tsirelson's upper bound (for $n \geq 2$), let

$m = n(n-1)/2$. Define a matrix G , with rows indexed by pairs ij , where $1 \leq i < j \leq n$, and columns indexed by the set $\{1, \dots, n\}$, as

$$G_{ij,k} = \begin{cases} 1/2m & i = k \\ -1/2m & j = k \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Then G defines an $m \times n$ XOR non-local game. Now look at the optimization problem (Γ) for this game. The objective function of (Γ) is invariant under conjugation of V by permutation matrices. Thus we can solve (Γ) by the standard technique of summing over the action of the symmetric group to reduce to a linear program. It follows that $V = (2m^2s^2)I + (1 - 2m^2s^2)E$ is a solution to (Γ) , where E is the matrix of all ones. The marginal row biases can be determined by the formula $c_{ij}^2 = (1 - V_{ij})/2m^2$. The solution algebra will be a Clifford algebra of rank $n-1$, and thus G requires $\lfloor (n-1)/2 \rfloor$ ebits.

7.3 Clifford strategies do not always use minimal entanglement

To give an example of a game for which the strategy of minimum entanglement is not Clifford, we use the graph construction of Proposition 7.1. Given an integer n , let Y_1, \dots, Y_n be the canonical generators of the Clifford algebra \mathcal{C}_n . Define a graph G whose vertices are ordered k -tuples (i_1, \dots, i_k) , where $1 \leq i_1 < \dots < i_k \leq n$, and k ranges between 1 and n . Place an edge in G between (i_1, \dots, i_k) and (j_1, \dots, j_l) if $Y_{i_1} \cdots Y_{i_k}$ anticommutes with $Y_{j_1} \cdots Y_{j_l}$ in \mathcal{C}_n . Let $CL(n)$ be the XOR non-local game A_G constructed from the graph G according to the recipe in Proposition 7.1. We want to show that the optimal strategies of minimum entanglement for $CL(n)$ are not Clifford.

Proposition 7.3. *Let (Γ) be the convex programming problem associated to $CL(n)$ by Proposition 3.3. For $n \geq 3$, every solution V to (Γ) has rank at least $(4n-8)/3$.*

Proof. We use a pigeon-hole argument. Let V be a solution to (Γ) , and take unit vectors $v(\tilde{i})$ in \mathbb{R}^N such that $V_{\tilde{i}, \tilde{j}} = v(\tilde{i}) \cdot v(\tilde{j})$. For convenience of notation we ignore order in the multi-indices, so for example $v(2, 1)$ would be the same as $v(1, 2)$. Now ij is an edge in G for every $i \neq j$, and hence $v(i) \cdot v(j) = 0$. Let $W = \text{span}\{v(i) : 1 \leq i \leq n\}$, and write

$$\text{span}\{v(\tilde{i})\} = W \oplus W^\perp.$$

It will also be convenient to let

$$W(i_1, \dots, i_k) = \text{span}\{v(i_1), \dots, v(i_k)\} \oplus W^\perp.$$

Now there are $\binom{n}{3}$ vectors of the form $v(ijk)$. If $l \notin \{i, j, k\}$ then $v(l) \cdot v(ijk) = 0$. Hence $v(ijk)$ is contained in $W(ijk)$, and has non-zero projection onto one of $W(ij)$, $W(ik)$, or $W(jk)$. For $i < j$ let Z_{ij} be the set of vectors $v(ijk)$ which have a non-zero projection onto $W(ij)$ (note that k need not be greater than i and j). There are $n-2$ vectors which could potentially be in Z_{ij} . However if $k \neq l$ then

$$(P_{W(ij)}v(ijk)) \cdot (P_{W(ij)}v(ijl)) = v(ijk) \cdot v(ijl) = 0,$$

where $P_{W(ij)}$ is the orthogonal projection onto $W(ij)$. Thus Z_{ij} contains no more than $\dim W(ij) = 2 + \dim W^\perp$ vectors. Thus we conclude that

$$\binom{n}{3} \leq (2 + \dim W^\perp) \binom{n}{2}.$$

Solving this inequality we find that $\dim W^\perp \geq \frac{n-8}{3}$, and hence

$$\dim \text{span}\{v(\tilde{i})\} \geq n + \frac{n-8}{3} = \frac{4n-8}{3}$$

□

Let \mathcal{A} be the solution algebra associated to $CL(6t+2)$. Every optimal vector solution to $CL(6t+2)$ has rank at least $8t$, so every Clifford representation uses at least $4t$ ebits. However, it is clear that any irreducible representation of \mathcal{C}_{6t+2} is also a representation of \mathcal{A} , since we can send X_{i_1, \dots, i_k} to $i^s Y_{i_1} \cdots Y_{i_k}$, where $s = \binom{k}{2}$. Thus $CL(6t+2)$ requires at most $3t+1$ ebits. It follows that any Clifford representation uses at least $t-1$ ebits more than what is required.

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